

CENTRALISERS OF C^∞ DIFFEOMORPHISMS

BY

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ABSTRACT. It is shown that if F is a hyperbolic contraction of R^n , coordinates may be chosen so that not only is F a polynomial mapping, but so is any diffeomorphism which commutes with F . This implies an identity principle for diffeomorphisms G_1 and G_2 commuting with an arbitrary Morse-Smale diffeomorphism F of a compact manifold M : if $G_1, G_2 \in Z(F)$, then $G_1 = G_2$ on an open subset of $M \Rightarrow G_1 \equiv G_2$ on M .

Finally it is shown that under a certain linearisability condition at the saddles of F , $Z(F)$ is in fact a Lie group in its induced topology.

Introduction. Let f be a C^∞ diffeomorphism of R^n onto itself which fixes the origin, and let $Df_0: R^n \rightarrow R^n$ be its first derivative at 0. We shall describe f as a *sink* on R^n if it is hyperbolic and a topological contraction: i.e., (i) every eigenvalue λ of Df_0 satisfies $|\lambda| < 1$ and (ii) $\bigcap_{n=0}^\infty f^n(U) = \{0\}$ for any bounded set U containing the origin. The k -jet of f , denoted f_k , is an element of $L^k(n)$, the group of k -jets of local diffeomorphisms of R^n which preserve the origin. We denote by $L^\infty(n)$ the group of formal power series; $L^\infty(n) = \text{inv lim } L^k(n)$. The theorem of K.-T. Chen and S. Sternberg [2], [7] applied to sinks, implies that there is k (computed from the eigenvalues of Df_0) such that the germ of f is conjugate by a C^∞ diffeomorphism germ g , to some L^k -conjugate of f_k , $\bar{f}_k$: thus

$$(I) \quad gfg^{-1} = \bar{f}_k.$$

We describe the normal form $\bar{f}_k$ in §2. Two sinks f and h are conjugate if and only if $\bar{f}_k$ and $\bar{h}_k$ are conjugate in $L^\infty(n)$.

It is our purpose to show that the space of diffeomorphisms commuting with f admits a finite dimensional parametrisation. The first main result is

THEOREM 1. *Let h be any local diffeomorphism of R^n such that $\bar{f}_k h = h \bar{f}_k$. Then $h = h_k$.*

An obvious corollary is that conjugating functions g in equation (I) are unique up to elements of the centraliser in $L^k(n)$ of $\bar{f}_k$.

The theorem is a generalisation of a corresponding theorem of N. Kopell [3] in which $\bar{f}_k = \bar{f}_1$, i.e., f is linearisable. Thus by an argument in her paper,

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there follows an identity principle for a sink, or for that matter, for a C^∞ Morse-Smale diffeomorphism of a compact C^∞ -manifold M without boundary. (For definitions, see [6].)

COROLLARY 1. *If g_1 and g_2 both commute with f , and if U is an open set of $R^n(M)$, then $g_1 = g_2$ on U implies $g_1 \equiv g_2$.*

A second corollary may be adduced which describes the space of solutions to the equation $[X, Y] = 0$, where X and Y are C^∞ vector fields on R^n , and the flow ϕ_t generated by Y consists of hyperbolic sinks.

COROLLARY 2. *Let Y be a vector field germ such that Y is an elementary contracting critical point: i.e., the eigenvalues μ of DZ_0 satisfy $\text{Re}(\mu) < 0$. Let $\mathfrak{Z}(Y) = \{X \mid [X, Y] = 0\}$. Then $\mathfrak{Z}(Y)$ is the Lie algebra of $Z(\bar{\phi}_1)$ the centraliser in $L^k(n)$ of $\bar{\phi}_1$ (k chosen as above).*

In order to prove the theorem, we give in §2 a normal form $\bar{f}_k$ for f ; it is related to the real Jordan form which the matrix f_k has in a certain faithful linear representation of $L^k(n)$. The jet $\bar{f}_k$ is an invariant of C^∞ -conjugacy of sinks. Indeed, using the representation one can give an alternative proof of the formal content of Sternberg's theorem.

The second main result is an extension of Theorem 1 to Morse-Smale diffeomorphisms. To a Morse-Smale diffeomorphism f , and to an orbit $\{x, f(x), \dots, f^m(x)\}$ in $\Omega(f)$, there is associated the spectrum of $D(f^m)_x$. At least one such orbit is a *sink*, for which all the eigenvalues of $D(f^m)_x$ have absolute value less than 1. A *source* for f is a sink for f^{-1} and any other point of $\Omega(f)$ is called a *saddle*.

Now let $\text{Diff}^\infty(M)$ denote the topological group of C^∞ diffeomorphisms of a compact manifold M , topologised by the C^∞ topology (see, for example [5]). $\text{Diff}^\infty(M)$ contains $Z(f) = \{g \in \text{Diff}^\infty(M) \mid gf = fg\}$ as a closed subgroup.

THEOREM 2. *Suppose f is a Morse-Smale diffeomorphism such that at each saddle y ,*

$$(II) \quad \lambda_i \neq \prod_{j=1}^n \lambda_j^{m_j},$$

where if m is the period of f on y , λ_i are the eigenvalues of $D(f^m)_y$, m_j are nonnegative integers, and there is $j \neq i$ with $m_j \neq 0$. Then as a topological group $Z(f)$ is equivalent to a Lie group.

In a subsequent paper we prove that generically $Z(f)$ is in fact discrete; such f cannot, for example, be a time-one map for a C^∞ vector field.

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1. **Preliminaries.** We shall adopt some notations of [4], which we briefly recall. $C^\infty(R^n)$ denotes the local ring of C^∞ mapping germs at the origin and M its maximal ideal. Analogously, let N be the ideal in $R[x_1, \dots, x_n]$ of polynomials without constant term. N is generated as a real vector space by the monomials—we denote the monomial $x_1^{i_1} \dots x_n^{i_n}$ by x^I , where $I = (i_1, \dots, i_n)$ is an ordered n -tuple of nonnegative integers. For such I , $I! = i_1! \dots i_n!$ and $\deg I = i_1 + \dots + i_n$. We shall need the fact that there is a canonical identification of N with $\sum_{i=1}^\infty O^i V$ (where $O^i V$ denotes the i th symmetric power of $V \approx R^n$) under which monomials in x are identified with monomials in the standard basis of V .

The Taylor homomorphism $j_k: M \rightarrow N$ is defined by

$$j_k(f) = \sum \frac{1}{I!} \frac{\partial^{\deg I} f}{\partial x^I}(0) x^I,$$

the sum being over all I with $1 \leq \deg I \leq k$. As is well known, j_k induces an isomorphism $M/M^{k+1} \approx N/N^{k+1}$, this quotient being denoted $J^k(n, 1)$, the ring of k -jets of vanishing real C^∞ functions. Having introduced N , we may speak of “complex k -jets”: these are elements of N/N^{k+1} when N is the ideal of complex polynomials in n -variables without constant term. $J^k(n, n) = J^k(n, 1) \otimes R^n$ is the space of k -jets of functions R^n to R^n and $L^k(n) \subset J^k(n, n)$ the group of jets invertible for the operation of composition.

There is a right linear action of $L^k(n)$ on $J^k(n, 1)$ defined by $x^I \cdot g = j_k(g^I)$ where $g = (g_1, \dots, g_n)$ is in $L^k(n)$. In the basis of monomials, g has the matrix G_{IJ} = coefficient of g^J on x^I . Hence by sending g to the adjoint of the linear map G_{IJ} , there is obtained a representation $\rho_k: L^k(n) \rightarrow GL(J^k(n, 1))$. The matrix of $\rho_k(g)$, in the basis of monomials, is G_{IJ} = coefficient of g^J on x^I . The indices I, J range over all monomials in n variables of degree not greater than k .

The following facts are easy to see.

- (i) ρ_k is faithful (and presents $L^k(n)$ as an algebraic group).
- (ii) If $\deg I = 1 = \deg J$, $(\rho_k(g))_{IJ}$ is the Jacobian matrix of g , $j_1(g) = Dg_0$. Moreover, if $1 < r \leq k$ and $\deg I = r = \deg J$, $(\rho_k(g))_{IJ}$ is the matrix of $Dg \circ \dots \circ Dg: O^r V \rightarrow O^r V$ (in the basis of monomials). This implies that the eigenvalues of $\rho_k(g)$ are all monomials λ^I of degree $\leq k$ in the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of Dg_0 .
- (iii) If f is a diffeomorphism of R^n , and if $\deg I = 1$, $\deg J = r \leq k$, then $\rho_k(f_k)_{IJ}$ is the matrix of $D^r f: O^r V \rightarrow V$ taken in the basis $\{J! x^J | \deg J = r\}$.

2. **Normal forms of diffeomorphisms. 1-jets.** Let A be a real linear operator. Then there is a basis $\{z_1, \bar{z}_1, \dots, z_r, \bar{z}_r, w_{2r+1}, \dots, w_n\}$ of C^n in which

the matrix of A is in Jordan form: if z_i is a (generalised) eigenvector for λ , so is $\bar{z}_i$ for $\bar{\lambda}$, and w_i are the real eigenvectors. By the real Jordan form of A , we mean the matrix which A takes in the basis $\{\operatorname{Re}(z_1), \operatorname{Im}(z_1), \dots, \operatorname{Im}(z_r), w_{2r+1}, \dots, w_n\}$.

EXAMPLE.

$$\begin{pmatrix} \operatorname{Re} \lambda & \operatorname{Im} \lambda & 1 & 0 \\ -\operatorname{Im} \lambda & \operatorname{Re} \lambda & 0 & 1 \\ 0 & 0 & \operatorname{Re} \lambda & \operatorname{Im} \lambda \\ 0 & 0 & -\operatorname{Im} \lambda & \operatorname{Re} \lambda \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \bar{\lambda} & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \bar{\lambda} \end{pmatrix}.$$

Notice that there is an involution $z_i \leftrightarrow \bar{z}_i$ of the basis of C^n which induces a corresponding involution σ of the monomials, which we denote by $z^I \leftrightarrow z^{\sigma(I)}$. If

$$z^I = z_1^{i_1} \bar{z}_1^{j_1} \dots z_r^{i_r} \bar{z}_r^{j_r} w, \quad z^{\sigma(I)} = z_1^{j_1} \bar{z}_1^{i_1} \dots \bar{z}_r^{i_r} w.$$

Complex jets. A complex k -jet or formal power series (∞ -jet) F is in normal form providing

- (i) $\rho_1(F_1)$ is in Jordan normal form,
- (ii) $\rho_r(F_r)$ is upper triangular for $k \geq r \geq 1$,
- (iii) $(\rho_r(F_r))_{IJ} \neq 0 \Rightarrow \lambda^I = \lambda^J$.

Real jets. A real k -jet or formal power series F is in normal form providing:

- (i) $\rho_1(F_1)$ is in real Jordan normal form, so there is a matrix A_1 , being a sum of blocks $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ so that $A_1 \rho_1(F_1) A_1^{-1}$ = Jordan form of $\rho_1(F_1)$.
- (ii) The complex power series $A_1 F_r A_1^{-1}$ is in normal form for each $k \geq r \geq 1$.

It follows from (ii) that $(A_k)_{IJ} = \overline{(A_k)_{\sigma(I)\sigma(J)}}$, and conversely, if this condition is satisfied for a complex formal power series or jet A_k , the eigenvalues of whose linear part occur in conjugate pairs, then A_k is derived from a real f.p.s. via conjugation by A_1 (see [1]).

A diffeomorphism germ is in normal form if its associated formal power series is in real normal form.

REMARKS. (1) The normal form is a conjugacy invariant of formal power series or jets, or diffeomorphisms, but it need not be unique. For instance, if $\lambda \neq 1$ is real, the 3-jets

$$\begin{aligned} F(x, y, z) &= (\lambda x, \lambda^2 y + x^2, \lambda^3 z + xy + x^3), \\ G(x, y, z) &= (\lambda x, \lambda^2 y + x^2, \lambda^3 z + xy) \end{aligned}$$

are distinct normal forms which are conjugate in $L^3(3)$, by $I + (0, x^2, 0)$.

(2) The matrix $\rho_k(F_\infty)$ will in general not be in Jordan form for $k > 1$. The following lemma is a generalisation of a lemma of Sternberg [7].

LEMMA. *For any (real) invertible FPS F_∞ there is a (real) invertible FPS G with $GF_\infty G^{-1}$ in (real) normal form.*

PROOF. We construct $G = \varprojlim G_k$ by induction on k . If $k = 1$, this is the familiar real Jordan normal form theorem for the Jacobian F_1 . For the inductive step, we observe that if $\rho_k(F_k)$ is in (real) normal form, so is $\rho_{k+1}(F_k)$. (Any entry $\langle f^I, x^J \rangle$ of $\rho_{k+1}(F_k)$ satisfies $\langle f^I, x^J \rangle = \sum \langle f^{I_1}, x^{J_1} \rangle \langle f^{I_2}, x^{J_2} \rangle$ where the sum is over all monomials $f^{I_1} f^{I_2} = f^I$ and $x^{J_1} x^{J_2} = x^J$. By inductive hypothesis, some summand can be nonzero only when $\lambda^{I_1} = \lambda^{J_1}$ and $\lambda^{I_2} = \lambda^{J_2}$, in which event $\lambda^I = \lambda^J$. As before, λ^I denotes a monomial in the eigenvalues of F_1 .)

It follows that if G_k is chosen so that $\rho_k(G_k F_k G_k^{-1})$ is in (real) normal form, then $\rho_{k+1}(G_k F_{k+1} G_k^{-1})$ will be in (real) normal form except possibly for entries f_{iJ} with $\deg i = 1$, $\deg J = k + 1$. (These entries represent the contribution of $D^k F$: $O^k V \rightarrow V$.) Dividing up

$$J^{k+1}(n, 1) = \frac{M}{M^2} \oplus \sum_{1 < r < k+1} \frac{M^r}{M^{r+1}} \oplus \frac{M^{k+1}}{M^{k+2}}$$

we see that $\rho_{k+1}(F_\infty)$ has, in this decomposition, the form

$$X = \begin{bmatrix} A & B & D \\ 0 & C & E \\ 0 & 0 & F \end{bmatrix},$$

where

$$\begin{pmatrix} A & B & 0 \\ 0 & C & E \\ 0 & 0 & F \end{pmatrix}$$

is in normal form. A is the Jacobian and F is its symmetric power of degree $(k + 1)$. Conjugating X by a matrix

$$Y = \begin{bmatrix} I & 0 & G \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we obtain $D - AG + GF$ in the top right-hand corner.

To prove the lemma is to solve $(D - AG + GF)_{iJ} = 0$ when $\lambda^I \neq \lambda^J$; i.e., $D_{iJ} = \sum A_{ij} G_{jK} - \sum G_{iK} F_{KJ}$ with $1 \leq j \leq n$ and $\deg K = k + 1$.

Using the assumption that $\rho_{k+1}(F_k)$ is upper triangular (as a complex matrix), this equation becomes

$$(*) \quad D_{iJ} = G_{iJ}(A_{ii} - F_{JJ}) + \sum_{i < j} A_{ij}G_{jJ} - \sum_{K < J} G_{iK}F_{KJ}.$$

Supposing G_{iJ} is known for $j < i$, G_{iK} for $K < J$, we may solve inductively for G_{iJ} when $A_{ii} \neq F_{JJ}$ (i.e. $\lambda^i \neq \lambda^J$).

To prove the statement for real jets, we must know that the basis change taking F_1 into its real Jordan form takes some real jet into Y . According to Birkhoff [1], the condition that this be so is that $\langle y^I, x^J \rangle = \overline{\langle y^{\sigma(I)}, x^{\sigma(J)} \rangle}$, where the involution σ on monomials is defined as above. It is clear in equation (*) that if D, A, F satisfy the condition, G will satisfy it by the same induction when $\lambda^i \neq \lambda^J$; and otherwise the choice of $G_{iJ} = 0$ also satisfies the condition.

REMARKS. (1) The lemma above constitutes the formal content of the theorem of Chen and Sternberg as in [2], [7] and [8], in the sense that by choosing a diffeomorphism with G_∞ as above, we may conjugate F by G to a diffeomorphism whose jet is in normal form. The analytic content is then to prove that two C^∞ hyperbolic diffeomorphisms with the same normal form are C^∞ -conjugate.

(2) If $\lambda^i \neq \lambda^J$ for any monomial λ^I of degree > 1 , the normal form is linear, whereupon F is C^∞ conjugate to a linear map.

(3) If F is a sink (source), there is k such that $\lambda^I = \lambda^J \Rightarrow \deg I \leq k$, so that the normal form of F is a k -jet, to which F is C^∞ -conjugate.

3. Centralisers. The crucial observation of this section is that if $L: V \rightarrow V$ is a linear map, and if $V = \Sigma V_i$ is its decomposition into generalised eigenspaces, then if $LM = ML$, each V_i is M -invariant. Applying this fact to $\rho_k(F_k)$ in normal form, if G_k is a commuting jet, then $\rho_k(G_k)$ also has the property $\rho_k(G_k)_{IJ} \neq 0 \Rightarrow \lambda^I = \lambda^J$; although of course it need not be upper triangular. Furthermore, if F_k is a contraction and k is not less than the maximal degree on the right-hand side of relations $\lambda^i = \lambda^J$, then if G_∞ commutes, $G_\infty = G_k = G_{k+r}$ for all r . In particular, F_k^{-1} (or G_k^{-1}) coincides with the inverse of F_k in $L^\infty(n)$, and so the polynomial mappings $F_k, F_k^{-1}, G_k, G_k^{-1}$ are globally defined diffeomorphisms of R^n .

The theorem is an application of the following

LEMMA. Let F_k be a contracting jet in normal form with k as above. Considered as a polynomial diffeomorphism of R^n , for x, y in a compact $K \subset R^n$, we have

$$(1) \quad |F_k^{-m}(x) - F_k^{-m}(y)| \leq p(m)\lambda^{km}|x - y|,$$

$$(2) \quad |F_k^m(x) - F_k^m(y)| \leq q(m)\mu^{km}|x - y|,$$

where p, q are polynomials; λ^{-1} = smallest eigenvalue of F_1 , μ = largest eigenvalue of F_1 .

PROOF. We prove (1). Let $\alpha: V \rightarrow S^k V$ be defined by $x \rightarrow x^I$; i.e., if $\{e_j\}$ is a basis of V and the symmetric products of these vectors are denoted e^I , then the coefficient of $\alpha(x)$ on e^I is x^I . Clearly α has a uniform Lipschitz constant on K .

If π is the projection $S^k V \rightarrow V$, then $F_k^m(x) = \pi \circ \rho(F_k)^m \circ \alpha(x)$ for any $m \in \mathbb{Z}$, because F_k is in normal form. Hence

$$\begin{aligned} |\pi \circ \rho(F_k^{-m}) \circ \alpha(x) - \pi \circ \rho(F_k^{-m}) \circ \alpha(y)| &\leq |\pi| |\rho(F_k^{-m})| |\alpha(x) - \alpha(y)| \\ &\leq \text{const} |\rho(F_k)^{-m}| |x - y|. \end{aligned}$$

We may write $\rho(F_k) = SU$ where these two linear maps commute, S is semisimple—we may suppose diagonal—and U is unipotent. Then $\rho(F_k)^{-m} = S^{-m} U^{-m}$ and

$$|\rho(F_k)^{-m}| \leq |S^{-m}| |U^{-m}| \leq \lambda^{km} p(m).$$

REMARK. $(\deg p) + 1$ = nilpotence degree of $(U - I)$.

PROOF OF THEOREM 1. According to the observation at the head of this section, with k as above, if g commutes with F_k , then for all $r \leq \infty$, $\rho_{k+r}(g) = \rho_k(g)$. Let g_k be the polynomial diffeomorphism of degree k such that $\rho_k(g_k) = \rho_k(g)$, then $g_k F_k = F_k g_k$, so that we may write $g g_k^{-1} = I + h$, commuting with F_k , and such that $h(x) = O(x^r)$ on a small enough neighbourhood U_r of 0 (for any r). In fact, $h \equiv 0$. Since $h(x) = (I + h - I)(x)$, we have

$$\begin{aligned} |h(x)| &\leq |F_k^{-m}(I + h)F_k^m - F_k^{-m}F_k^m(x)| \leq p(m)\lambda^{km} |(I + h)F_k^m(x) - F_k^m(x)| \\ &\leq p(m)\lambda^{km} |h(F_k^m(x))| \leq p(m)\lambda^{km} |F_k^m(x)|^r \\ &\leq \text{const } p(m)\lambda^{km} q(m)^r \mu^{kmr} |x|^r. \end{aligned}$$

Choosing r and U_r such that $\lambda\mu^r < 1$, and taking the limit of the right-hand side as $m \rightarrow \infty$, we see $h(x) \equiv 0$, because the exponential convergence of $(\lambda\mu^r)^{km}$ dominates the polynomial convergence of $p(m)q(m)^r$.

4. Centralisers of Morse-Smale diffeomorphisms. In this section we prove Theorem 2, stated in the Introduction. If p is the number of points in $\Omega(f)$, since $Z(f)$ acts on $\Omega(f)$, there is a homomorphism from $Z(f)$ to S_p whose kernel contains the identity component $Z(f)_0$: we show $Z(f)_0$ is a Lie group. Moreover, we may suppose $\Omega(f)$ consists of fixed points because $Z(f) \subset Z(f^r)$ for any r .

If $\{S_i\}$ are the sources in $\Omega(f)$ and $\{N_j\}$ the sinks, we may choose coordinates for their unstable and stable manifolds, $w^u(S_i)$ and $w^s(N_j)$, so that f is in normal form, and there are Lie groups $G_i = Z(f|_{w^u(S_i)})$, $G_j = Z(f|_{w^s(N_j)})$. Then

if $g \in Z(f)$ acts trivially on $\Omega(f)$, g leaves invariant the $w^u(S_i)$ and $w^s(N_j)$, and so $g|_{w^u(S_i)} \in G_i$, $g|_{w^s(N_j)} \in G_j$. Hence there is defined a homomorphism $R: Z(f)_0 \rightarrow \prod_{i,j} G_i \times G_j$.

By the identity principle this is injective, and it is continuous for the C^∞ topology on $Z(f)_0$. The conclusion will follow from the fact that R is a closed map. In other words, if g_m are C^∞ diffeomorphisms in $Z(f)_0$, such that $g_m \rightarrow g$ C^∞ -uniformly on compacta in $\bigcup w^u(S_i) \cup w^s(N_j) = M \sim \{\text{saddles}\}$, then the convergence is in fact uniform on M ; in particular, g is C^∞ and hence in the image of R .

We restrict attention to a saddle, about which, by assumption (see [8]), coordinates may be chosen making f linear. In this coordinate system, there are unique f -invariant linear subspaces E^u and E^s which span R^n ; the expanding and contracting eigenspaces of the (now linear) map f . The diffeomorphisms g_m , hence the map g , leave these subspaces invariant, whereupon from Theorem 1, $g_m^u = g_m|_{E^u}$ and $g_m^s = g_m|_{E^s}$ are already uniformly convergent on compacta in $E^u \cup E^s$; it follows that $g|_{E^u \cup E^s}$ is C^∞ .

Since f is linear, $f = f^u \times f^s: E^u \times E^s \rightarrow E^u \times E^s$. This means f commutes with each $g_m^u \times g_m^s$, and therefore with $g_m \circ (g_m^u \times g_m^s)^{-1}$. This sequence is the identity along $E^u \cup E^s$, and converges uniformly on all compacta to g if and only if $g_m \rightarrow g$ uniformly on all compacta. Hence, we may make the simplifying assumption that $g_m = g = I$ on $E^u \cup E^s$.

The first step is to show uniform C^0 -convergence. Let $g_m = S_m + U_m$ be the coordinate expansion of g_m ; thus, $S_m(U_m): R^n \rightarrow E^s(E^u)$. Then $f^{-r}U_m f^r = U_m$ and $f^t S_m f^{-t} = S_m$ for every $r, t \in \mathbb{Z}$ and every m , by the linear and hyperbolic properties of f . Consequently, in any norm

$$|g_m(x) - g(x)| \leq |f^{-r}(U_m f^r(x) - U f^r(x))| + |f^t(S_m f^{-t}(x) - S f^{-t}(x))|.$$

If $W \subset R^n - \{0\}$ is a compact set containing fundamental regions for f^u and f^s , then by uniform convergence on W , for high m , $|U_m - U| + |S_m - S| < \epsilon$ on W . But for x in any compact neighborhood of zero, there are numbers r, t so that $f^r(x)$ and $f^{-t}(x)$ are in W . The result follows from the fact that f^{-r} acts contractively on E^u and f^s contractively on E^s .

The argument for convergence of the higher derivatives is more delicate: we observe that $D^k f = 0$ for $k \geq 2$ (f is linear). Then by applying the chain rule ($f = Df$ is a constant linear map)

$$(**) \quad (Df)^r D^k g_m(x) \underbrace{(Df \times \dots \times Df)^{-r}}_k = D^k g_m(f^r x)$$

for all k, r, m, x . This equation is in fact the r th iteration of the linear operator $F(A) = Df \circ A \circ (Df \times \dots \times Df)^{-1}$ operating on L_{sym}^k the space of symmetric

k -multilinear maps. If $L_{\text{sym}}^k = L^u \oplus L^s \oplus L^c$ is the decomposition into eigenspaces for F for the eigenvalues of absolute value > 1 , < 1 , and $= 1$ respectively, then the argument used in the C^0 case above may be applied to show that the L^u and L^s components of $D^k G_m - D^k G$ tend to zero as $x \rightarrow 0$. It remains to settle the component on L^c . Note that for $F|_{L^c}$, with a norm on L_{sym}^k , $|F^r| \leq p(r)$ for some polynomial p .

We claim that G_m has infinite contact with the identity along $E^u \cup E^s$. If $x \in E^s$, we see that $F^r(D^k g_m^c(x)) \rightarrow D^k g_m^c(0)$ as $r \rightarrow \infty$. For a linear map, this can only happen for a fixed point, and so $D^k g_m^c$ is constant along E^s (similarly, E^u , using F^{-1}). In fact, for every $k > 1$, $D^k g_m^c(0) = 0$, since, any jet J_k commuting with the linear map $f = Df_0$, satisfies $J_k = J_1$ by the assumption $\lambda_i \neq \Pi \lambda_j^{n_j}$. But $g_m = I$ along E^u and E^s (by the simplifying assumption above), and this implies that $J_1 = I$: this proves the claim.

We can now show that (g_m) is uniformly convergent on compacta. By (**), for the L^c component, we have

$$\begin{aligned} |D^k g_m(x) - D^k g(x)| &= |F^{-r}(D^k g_m(f^r(x))) - F^{-r}(D^k g(f^r(x)))| \\ &\leq p(r)|D^k g_m(f^r(x)) - D^k g(f^r(x))| \\ &\leq p(r)(|D^k(g_m - g)(f^r(x)) - D^k(g_m - g)(z)| + |D^k(g_m - g)(z)|). \end{aligned}$$

For any x in a small compact neighborhood of 0, there is r such that $f^r(x) \in W$, then $z \in E^u$ may be chosen to minimise the distance between $f^r(x)$ and E^u . By the preceding, $D^k(g_m - g)(z) = 0$. Now applying the Mean Value Theorem, where $|A|_W$ denotes supremum on W ,

$$|D^k(g_m - g)(f^r(x)) - D^k(g_m - g)(z)| \leq |D^{k+1}(g_m - g)|_W |f^r(x) - z| \text{ const } \lambda^r$$

for some $0 < \lambda < 1$.

Since exponential convergence still dominates polynomial convergence, this completes the proof. g_m is for each k , C^k -uniformly convergent on compacta, R is a closed map, and g is a C^∞ diffeomorphism.

REFERENCES

1. G. D. Birkhoff, *Dynamical systems*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 9, Amer. Math. Soc., Providence, R. I., 1966, p. 62.
2. K.-T. Chen, *Local diffeomorphisms— C^∞ realization of formal properties*, Amer. J. Math. 87 (1965), 140–157. MR 30 #3484.
3. N. Kopell, *Commuting diffeomorphisms*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 165–184. MR 42 #5285.
4. H. I. Levine, *Singularities of differentiable mappings*, Proc. Liverpool Singularities Sympos., I, Lecture Notes in Math., vol. 192, Springer-Verlag, Berlin and New York, 1971, p. 6.
5. J. N. Mather, *Stability of C^∞ mappings. II: Infinitesimal stability implies stability*, Ann. of Math. (2) 89 (1969), 254–291. MR 41 #4582.

6. J. Palis and S. Smale, *Structural stability theorems*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 223–231. MR 42 #2505.
7. S. Sternberg, *Local contractions and a theorem of Poincaré*, Amer. J. Math. 79 (1957), 809–824. MR 20 #3335.
8. ———, *On the structure of local homeomorphisms of Euclidean n -space. II*, Amer. J. Math. 80 (1958), 623–631. MR 20 #3336.

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